Making Sense of Representations: Card Sorting

Why is slope so important? How can I write an equation that passes through these points? Why do we need to know three different equations for the same parabola?

These are just three of the questions students often ask that demonstrate their difficulty connecting different representations of relations and functions. However, making these connections is crucial if students are to make sense of mathematics and be able to reason and model by looking for repeated patterns and structure (NCTM 2000).

Card-sorting activities can be useful in helping students make sense of these connections because they ask students to analyze multiple representations of functions simultaneously (Chauvot and Benson 2008). Working collaboratively, students explain and justify their reasoning as they observe and form conjectures about the patterns of relations among representations of functions. By presenting their results and critiquing the results of others, students are able to detect and correct errors in the thinking of their peers. In this way, even incorrect observations or observations of patterns that are not typically addressed in class can lead to the development of deep mathematical understandings. Through this process, students participate in multiple mathematical practices, including making sense of problems, reasoning abstractly, constructing and critiquing arguments, and looking for and making use of structure (CCSSI 2010).

Two examples of card-sorting activities are described here. The key to these activities is that they allow students to form their own conclusions about the ways in which different representations of relations connect to one another. In other words, students construct their own mathematical knowledge. The role of the teacher, therefore, is to ask meaningful questions about students’ thinking rather than supply answers.

Each example below contains common student questions and conjectures as well as questions that teachers find helpful in guiding students through the activity. As you read the examples, consider what other questions your students may have as well as what questions you may want to ask students to help them clarify their thinking.

**ACTIVITY 1: TABLES AND GRAPHS**

Use card stock to print activity sheet 1; cut the cards apart and laminate them for future use. Print enough copies to ensure that when students use them in small groups, each group can receive one full set. A table of values is shown on each card (see fig. 1); the key to this exercise is that each table of values represents a linear equation with one of four y-intercepts (–4, –1, 2, or 5) and one of four slopes (–3/2, 3/2, –1/2, or 1/2). This activity helps students recognize similarities in representations of lines with the same y-intercept or slope.

**Engage**

Place students into small groups and give each group a set of cards. Direct students to look for patterns that can be
used to describe more than one table of values. After several minutes of observations, pick one group to share an observation. Direct the other groups to verify this observation on their own. If there is disagreement, guide students in talking about why they agree or disagree with the observation until the class reaches a consensus. This process should eliminate any incorrect observations as students point out counterexamples.

Once a consensus is reached, continue asking other groups to share an observation until all observations have been discussed. Some observations may be less specific than others. For example, some groups may notice that in several tables the $y$-values decrease by 3 for each row, whereas other groups may simply notice that in several tables the $y$-values decrease. Groups may also notice unanticipated patterns. This variety of responses is not only acceptable but also demonstrates students' creative use of mathematics. Regardless of what patterns students notice, acknowledge and record all observations that are verified by the class. Possible student observations about tables of values include the following:

- “All the tables have the same $x$-values listed.”
- “Almost all the tables list –1 as a $y$-value.”
- “Many tables contain the point $(0, 2)$.”
- “For some tables, the $y$-values increase; for others, the $y$-values decrease.”
- “The $x$-values go up by 2.”
- “The $y$-values change by either 1 or 3 for each table.”
- “The $y$-values start off positive for some tables and negative for others.”
- “Some tables only have positive $y$-values, but other tables have negative $y$-values too.”

**Explore**

Explain to students that they will be exploring the differences among tables of values. Ask students which observations are true for all tables and which observations are true for only some tables. Assign the observations that are true for only some tables to the groups. Be sure to assign each observation to at least one group; the same observation may be assigned to more than one group if there are not enough observations for all the groups.

Have the groups sort their cards into piles according to one of the observations they are given. For example, if a group is exploring an observation about the $y$-intercept, that group should categorize its cards according to the $y$-intercept; all tables of values with the same $y$-intercept should be placed together.

After all cards are sorted, instruct students to examine their largest category (or one of the largest categories, if they have piles of the same size). For the cards in this category, have students use a single set of axes to plot each table of values and connect the plotted points with a straight line. It may be helpful for students to use colored pencils and markers—and use the lettering associated with each table of values—during this step to keep track of which line represents which table of values.

**Explain**

Once all lines have been graphed, ask groups to make observations about the lines: What do they have in common? In what ways do they look similar? Have students complete the following statement using their observations of the tables of values and of the lines:

Conjecture: When tables of values [show this characteristic], then the graphs [share this characteristic].

It is likely that some groups will form their conjecture much more quickly than others, just as some groups will work with smaller piles of cards. For groups that finish early, challenge them to extend their conjecture by considering cards in another pile, not already examined when forming their hypothesis. For example, if a group makes a statement about tables of values that have a $y$-value of 3 when the $x$-value is 0, ask the students if they could make a similar statement for tables of values for which $y$ is 2 when $x$ is 0.

Once all groups have formed at least one conjecture, have each group report its findings to the class. This reporting process is the heart of the activity.

**Fig. 1** Two cards for activity 1 show tables of values.

Students in each group should explain one observation that they explored, how they sorted their cards, and the results of the sorting process. They should identify the group of cards they used, share the graphs they plotted and the conjecture they formed. Allowing students to display their cards and create graphs on a document camera helps everyone see and evaluate their work. They can also reference the letters given on each card to help their peers follow along.

Other groups should verify the results. Encourage students to participate by requiring each group to make a constructive comment or ask a clarifying question at the end of each presentation; this is an opportunity to correct errors in their peers’ thinking processes. If an explanation offered is unclear and students do not question it, then also ask clarifying questions. Examples of clarifying questions follow:

- “If the tables that have a $y$-value of 3 when $x$ equals 1 all have graphs that pass through $(1, 3)$, what does that mean for a table of values for which $y$ equals 4 when $x$ equals 1?”
- “What do you mean when you say that the graph goes up or goes down?”
- “I agree that all your lines decrease; some seem to look more alike than others. What could be the reason for this?”
- “You said that all your lines start out above the $x$-axis. Do they stay above the $x$-axis? How can we tell which ones will stay above the $x$-axis and
on the reverse (see fig. 2), by cutting the graph on one side and the equations cards, which can be laminated, showing representations of parabolas. Create includes several differ-

ACTIVITY 2: GRAPHS AND EQUATIONS

Activity sheet 2 includes several different representations of parabolas. Create cards, which can be laminated, showing the graph on one side and the equations on the reverse (see fig. 2), by cutting along solid lines and folding along dotted lines. For students unfamiliar with the three given versions of the equation of the parabola, this activity is a great introduction.

Occasionally, students may not have observed a discernable pattern in the graphs. This is a perfect opportunity to discuss the limitations of graphs, which may not easily illustrate every pattern in a linear relation.

Once all students in the class agree with the statements presented, they should record them in their notebooks, including those that resulted from unanticipated patterns. Although the results of some patterns may not be useful in future lessons, they represent students’ understandings about linear relations and should be valued.

At the end of this activity, check understanding by directing students to examine a specific table of values and asking them to write down what each hypothesis predicts about the graph of the line based on the table of values. Using these predictions, have students graph the line without plotting each point in the table of values.

Engage

Divide students into small groups and give each group a set of cards with the graphs facing up. Allow students to analyze the cards, looking for patterns and commonalities. After discussing observations within the groups, each group should report one observation. As in activity 1, the cards have been lettered to help students reference them. The other groups should verify each observation, as in activity 1.

If students disagree, ask them to explain their thinking. This is the opportunity for the class to eliminate any incorrect observations made. Acknowledge and record all student observations for which there is agreement. After a list of observations has been created, ask students whether there is any single characteristic that describes all the graphs. Whether they have included this characteristic in their list or not, students should be able to notice that all the graphs are parabolas. Ask students to focus on one parabola of their choosing and determine which of the listed observations may help them distinguish that parabola from other parabolas.

For each student volunteer, note on the whiteboard which parabola he or she focused on and check off those obser-
vations used to describe the parabola. This process should guide the class into a discussion of roots or x-intercepts, y-intercepts, and vertices. Questions that may guide students toward these characteristics follow:

- “How would you describe the shape of these graphs to someone?”
- “How do the shapes differ?”
- “If you wanted to tell someone how to draw this graph, what would you tell them to do first?”
- “Can we easily pick out the coordinates of any of the points on these graphs?”
- “Which points on the graph would you need to plot in order to accurately draw the graph?”
- “Is there another graph with the same vertex or same intercept as this one? How could we tell the difference between the two graphs?”

Explore

Once students are focused on the concepts of intercepts and vertices, ask each group to pick one of the class-generated and agreed-upon observations to explore further. Working in groups, students sort their cards into piles according to the chosen observation. After sorting, instruct students to flip over the cards, keeping them in separate piles. Then have students examine the different forms of the equation of the parabolas and look for similarities in one or more forms of the equations that are in the same pile.

For example, if groups choose to focus on the conditions under which parabolas open up or down, they should notice that all three forms of the equation begin with a negative value when the parabola opens down. On the other hand, if groups focused on the values of the x-intercepts, then they probably will detect similarities only in the factored form. Finally, some groups may have chosen a characteristic that is not illustrated easily in any of the forms (for example, the y-value when x = 2). If so, explain to the students that some observations do not lead to immediate generalizations but are still valid and worthy. (For example, equations written in the form $y = a(x - 2)^2 + b(x - 2) + c$ would relate to the observation of the y-value
when \( x = 2 \).) This realization may increase students’ willingness to persevere in further problems and activities.

**Explain**

Students should now use their observation of commonalities to hypothesize about the relationship between the characteristic chosen and the values of coefficients in one or more forms of the equation of the parabola. Students who did not observe commonalities should examine the cards in a second pile and determine whether any commonality exists in the equations for that set.

Some groups are likely to finish more quickly than others, depending on how many cards share the characteristic they examine. Encourage these groups to attempt to alter their conjecture to describe behavior of parabolas they did not examine. For example, if they examined the parabolas with a \( y \)-intercept of 6 and found that the constant in the standard form of the equation for these parabolas was always 6, have them hypothesize about the parabolas with a \( y \)-intercept of –6 or 9.

Once all groups have verified their result, have them share their analysis using a document camera. Encourage students to ask clarifying questions. If the groups have not explored all the characteristics of parabolas desired, students can examine the equations that are now faceup on their cards. Sorting the equations themselves by characteristics desired, students should record all equations they were not already discussed, students can form a clear understanding of how various representation are related at the beginning of the unit, the rest of the unit can focus on answering those important “Why?” questions while reinforcing the relationships that students observed and recorded during the card sort.

**Extending Card Sorting**

As these activities illustrate, card-sorting activities provide ample opportunities for critical inquiry and questioning of students’ understandings. Many other card-sorting variations for secondary school students already exist. Johnson’s “Conic Cards” (Gruen 2011) and “Sinusoidal Sort,” created by Hunter and Crawford (Hunter 2013), are just two examples. However, you may want to consider creating your own card-sorting activities to help address your students’ specific needs. By carefully choosing which functions to include in a deck and how to represent the functions, you can guide your students to reason about specific characteristics of functions and relationships among various representations of functions.

Regardless of which cards you choose to use, sorting activities can be powerful instruments that allow students to construct their own understanding of the relationships among different representations of a type of function. At the end of this activity, students should feel confident that they know how representations are related.

What remains, then, is understanding why these relationships exist. Card sorting is an excellent way to open a unit about a particular kind of function. If students can form a clear understanding of how various representation are related at the beginning of the unit, the rest of the unit can focus on answering those important “Why?” questions while reinforcing the relationships that students observed and recorded during the card sort.

**REFERENCES**


KATE MEREDITH RAYMOND, kate.m.raymond@ou.edu, is a mathematics curriculum writer for the K20 Center for Educational and Community Renewal at the University of Oklahoma in Norman. She is pursuing a PhD in mathematics education at the Jeannine Rainbolt College of Education at the University of Oklahoma.
### Activity Sheet 1: Tables of Values for Linear Relations Card Sort

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*From the December 2015/January 2016 issue of* Mathematics Teacher
Activity Sheet 2: Graphs and Equations of Parabolas

Cut along solid lines, fold along dotted lines, and glue or laminate to make two-sided cards. Two of the sixteen cards are shown here. For the remainder of the set, see the online article at http://www.nctm.org/mt.

A

\[ y = 3x^2 - 6x - 9 \]
\[ y = 3(x - 1)^2 - 12 \]
\[ y = 3(x - 3)(x + 1) \]

P

\[ y = -3x^2 + 12x - 9 \]
\[ y = -3(x - 2)^2 + 3 \]
\[ y = -3(x - 3)(x - 1) \]
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